TURBULENCE
in Fluids and Plasmas

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Outline

- General features and Kolmogorov's theory
- 2-D and quasi-2-D limits of turbulence
- Examples of fluid and plasma 2-D turbulence
- Inverse cascade and formation of structures – Large scale flows
- Intermittency and non-Kolmogorov's turbulence
- Non-linear dynamics

General features and Kolmogorov's theory

Fluid motions

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \upsilon \nabla^2 \mathbf{v} \; ; \quad \nabla \cdot \mathbf{v} = 0$$

Laminar flows,
regular, deterministic description

Kinetic energy of flows dies out due to fluid viscosity

$\mathrm{Re} \geq 1$

Turbulent flows

a)irregularity - statistical treatment
b)rapid mixing - turbulent diffusion
c)wide range of length scales

$\mathrm{Re} >> 1$

Reynolds number $\Rightarrow \quad \mathrm{Re} = \frac{\mathrm{v}_0 L_0}{\upsilon}$

Dissipation of kinetic energy of turbulent flow?

Richardson's idea (Landau-Hopf mechanism)

1)Turbulence causes the formation of eddies of many different length-scale structures.
2)Most of the kinetic energy of a turbulent motion is contained in the large-scale structures.
3)The energy "cascades" from these large-scale structures to smaller scale structures by an inertial and essentially inviscid mechanism.
4)This process creates structures that are small enough that molecular diffusion becomes important and viscous dissipation of energy finally takes place.

$$\underbrace{L_0 \geq L_1 > L_2 > L_3 > \ldots > L_n > ..}_{\hat{i}nertial\ \ interval, \quad \varepsilon = const} \qquad . > L_N$$

Kolmogorov scales, dissipation of the energy by viscosity

$$\rightarrow \ direction\ of\ \ the\ energy\ cascade,\ \varepsilon \rightarrow$$

This is so called **direct** energy cascade

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \upsilon\nabla^2\mathbf{v} \; ; \quad \nabla\cdot\mathbf{v} = 0$$

$$\textit{Reynolds number} \quad \Rightarrow \quad \mathrm{Re}_0 = \frac{\textit{inertial term}}{\textit{viscous term}} \equiv \frac{\mathrm{v}_0 L_0}{\upsilon}$$

$$\underbrace{L_0 \geq L_1 > L_2 > L_3 > \ldots > L_n > ..}_{\hat{\imath}\textit{nertial interval}, \quad \varepsilon = const} \qquad . > L_N$$

Kolmogorov scales, dissipation of the energy by viscosity

$$\underbrace{\mathrm{Re}_0 \geq \mathrm{Re}_1 > \mathrm{Re}_2 > \mathrm{Re}_3 > \ldots > \mathrm{Re}_n > \ldots}_{\hat{\imath}\textit{nertial interval}, \quad \varepsilon = const} \qquad \underbrace{\ldots > \mathrm{Re}_N}_{\substack{\textit{Kolmogorov scales,} \\ \textit{dissipation of the energy} \\ \textit{by viscosity}}}$$

Kolmogorov's hypotheses - turbulence is locally isotropic

1) Energy dissipated per unit time and per unit mass by viscosity

$$\varepsilon \propto \frac{v_0^3}{L_0} \quad \Rightarrow \quad \begin{array}{ccc} v_N & \sim & v_0 \,\mathrm{Re}_0^{-1/4} \\ L_N & \sim & L_0 \,\mathrm{Re}_0^{-3/4} \end{array}$$

$\Leftrightarrow$ *Kolmogorov's scales*

2)Inertial interval: turbulence characteristics are specified in terms of $\varepsilon = const$

$$\left\langle \left(v_1 - v_2\right)^2 \right\rangle \equiv \left\langle \left(\delta v\right)^2 \right\rangle \sim \varepsilon^{2/3} r^{2/3}, \quad r \sim L_n$$

$$E\left(k\right) \sim \varepsilon^{2/3} k^{-5/3}, \qquad L_0^{-1} << k << L_N^{-1}$$

$\Leftrightarrow$ *Kolmogorov's turbulent spectra*

2-D and quasi-2-D limits of turbulence

2-D Fluid turbulence

$$\mathbf{v} = -\left(\nabla\phi \times \mathbf{z}\right)$$
$$\nabla \times \mathbf{v} = \nabla^2\phi\mathbf{z} \quad \Rightarrow \quad \frac{\partial \nabla^2\phi}{\partial t} - \left(\nabla\phi \times \mathbf{z}\right)\cdot\nabla\nabla^2\phi = 0$$
$$\omega_k = 0$$

Inertial interval - Invariants

$$W = \int\left(\nabla\phi\right)^2 d\mathbf{r} \quad \Leftrightarrow \quad \textit{energy}$$
$$V = \int\left(\nabla^2\phi\right)^2 d\mathbf{r} \quad \Leftrightarrow \quad \textit{enstrophy}$$

Energy cascades into longer scales =
Inverse cascade process in **k**-space

Arnold, 1965; Kraichnan, 1967

quasi-2-D limits of turbulence

Inhomogeneous media

admits excitation of linear waves $(\omega_k \neq 0)$ *with characteristic spatial scales*

Fluid, rotating atmosphere

Rossby waves

$$\rho_g = (gH_0)^{1/2} / \langle f \rangle$$

Rossby radius

$$\varepsilon = \frac{1}{\langle f \rangle} \frac{\partial}{\partial t} \approx \left| \rho_g \nabla \ln \frac{H_0}{f} \right| << 1$$

Magnetized plasma

Drift type waves

$$\rho_S = (T_e / m_i)^{1/2} / \omega_{ci}$$

ion Larmor radius

$$\varepsilon = \frac{1}{\omega_{ci}} \frac{\partial}{\partial t} \approx \left| \rho_S \nabla \ln \frac{n_0}{B_0} \right| << 1$$

admits **excitation** *of linear waves*
$(\omega_k \neq 0)$ *with characteristic spatial scales*

Rossby waves

Drift - type waves

$$\rho_g = (gH_0)^{1/2} / \langle f \rangle$$

Rossby radius

$$\rho_S = (T_e / m_i)^{1/2} / \omega_{ci}$$

ion Larmor radius

$$\varepsilon = \frac{1}{\langle f \rangle} \frac{\partial}{\partial t} \approx \left| \rho_g \nabla \ln \frac{H_0}{f} \right| << 1$$

$$\varepsilon = \frac{1}{\omega_{ci}} \frac{\partial}{\partial t} \approx \left| \rho_S \nabla \ln \frac{n_0}{B_0} \right| << 1$$

$$\omega_R = \left[(\mathbf{k} \times \mathbf{z}) \cdot \nabla \ln f \right] / \left(1 + k^2\right)$$

$$\omega_R = \left[(\mathbf{k} \times \mathbf{z}) \cdot \nabla \ln n_0 \right] / \left(1 + k^2\right)$$

wave turbulence $\Rightarrow$ *energy cascade in* $k -$ *space?*

Inverse cascade and formation of structures – Large scale flows

Quasi-2-D wave turbulence

$$\left.\begin{aligned}&\mathbf{v} = -(\nabla\phi\times\mathbf{z})\\&\nabla\times\mathbf{v} = \nabla^2\phi\mathbf{z}\\&\omega_{\mathbf{k}}(\mathbf{k})\neq 0\end{aligned}\right\} \Rightarrow \frac{\partial}{\partial t}\left(\nabla^2\phi-\phi\right)-\left[(\nabla\phi\times\mathbf{z})\cdot\nabla\right]\left(\nabla^2\phi+\ln\frac{\omega_{ci}}{n_0}\right)=0$$

Charny-Obukhov-Hasegawa-Mima equation

$$W = \int\left[(\nabla\phi)^2+\phi^2\right]d\mathbf{r} \quad \Leftrightarrow \quad \textit{energy}$$

$$V = \int\left[\left(\nabla^2\phi\right)^2+(\nabla\phi)^2\right]d\mathbf{r} \quad \Leftrightarrow \quad \textit{enstrophy}$$

Inertial interval –
energy cascading in k-space?

Inertial interval - spectral properties

Quasi-2-D **wave** turbulence

$$\omega_{\mathbf{k}} = f(\mathbf{k}) \neq 0$$

turbulence property of H-M eq. - cascading **model** in inertial interval

Decay of one wave into two other waves:

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$$

k-values of two waves produced by the decay proces can **trace** the cascade proces

$$\left.\begin{aligned} energy &\equiv W_{\mathbf{k}} \\ enstrophy &\equiv V_{\mathbf{k}} \end{aligned}\right\} = f(\mathbf{k})$$

$\Leftrightarrow$

Thus, **k**-values decide the partition of energy and enstrophy of newly created waves.
The cascade process therefore gives the energy and enstrophy transfer in **k**-space.

Inertial interval - spectral properties

cascading model produces energy spectrum with **two** distinctive regions $W_{\mathbf{k}} = f(\mathbf{k})$

One, at $k >> k_c$
spectrum is isotropic with $W_{\mathbf{k}} \sim k^{-4}$
and turbulence is self-similar $\equiv$
Direct cascade in 2-D fluid turbulence

Second, at $k << k_c$

Inverse cascade, Spectrtum is anisotropic with k_x and k_y , turbulence is
not self-similar

Quasi-2-D **wave** turbulence is in general **not self-similar** and characterized by the **anisotropic** spectrum

two distinctive regions

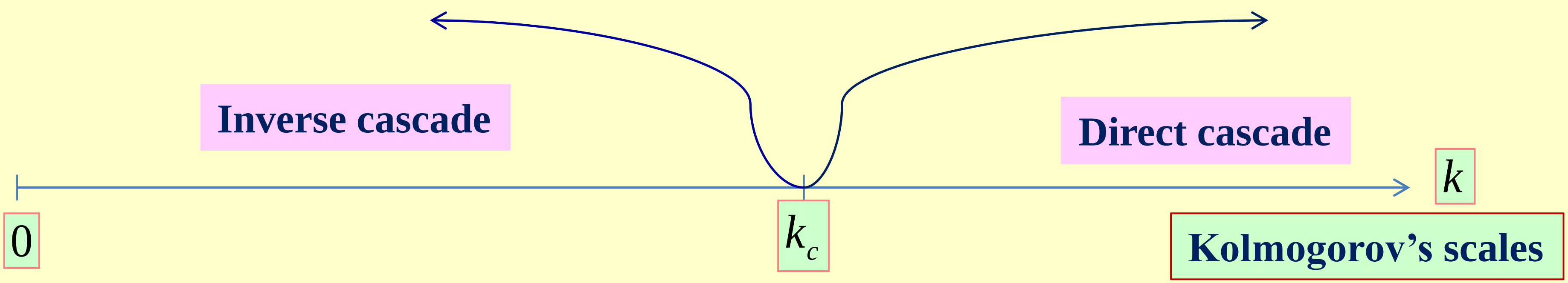
Second, at $k \ll k_c$

Inverse cascade, Spectrtum is anisotropic with k_x and k_y , turbulence is

not self-similar

One, at $k \gg k_c$
spectrum is isotropic with $W_{\mathbf{k}} \sim k^{-4}$
and turbulence is self-similar $\equiv$
Direct cascade in 2-D fluid turbulence

$$W_{\mathbf{k}} = f\left(\mathbf{k}\right)$$



Quasi-2-D **wave** turbulence is in general **not self-similar** and characterized by the **anisotropic** spectrum

Large scale flows, intermittency and non-Kolmlgorov's turbulence

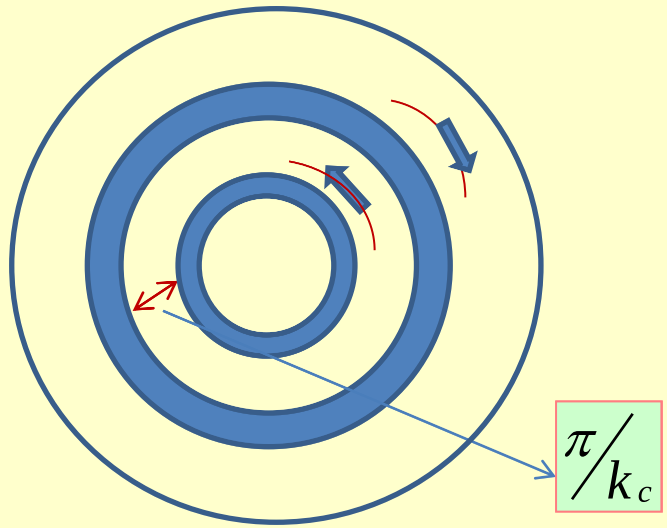
Spectrum condensation $\Leftrightarrow$ $\gamma_D(k_x, k_y) \Rightarrow 0 \Rightarrow \left\{ \begin{matrix} k_y \approx 0 \\ k_x \approx k_c \end{matrix} \right|, \; k_c \sim (1/L_0\phi_0)^{1/3}$

Inertial interval at $k << k_c$
Spectrtum is **anisotropic** and tends to **condense** at

$$\left. \begin{matrix} k_y \approx 0 \\ k_x \sim k_c \end{matrix} \right\}$$

This predicts the formation of **zonal flows** in *y*-direction which are periodic in the *x*-direction $\equiv$ **Coherent flow structure**

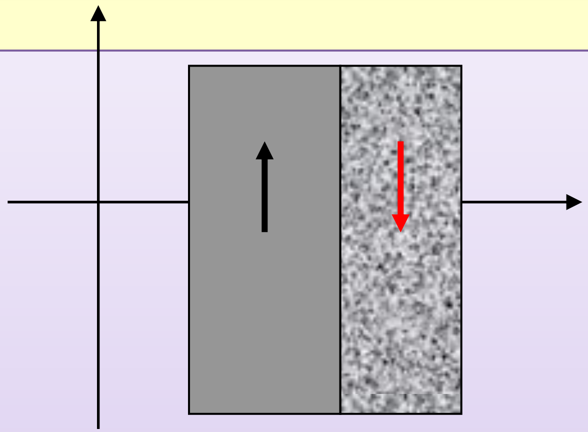
$\Leftrightarrow$



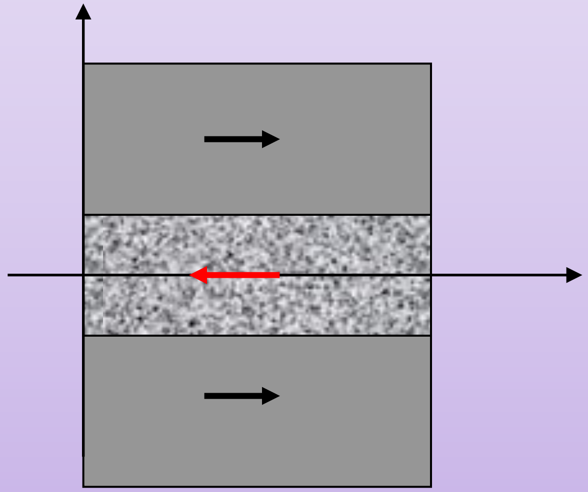
Quasi-2-D **wave** turbulence is dominated by such coherent structures, turbulence is **intermittent**

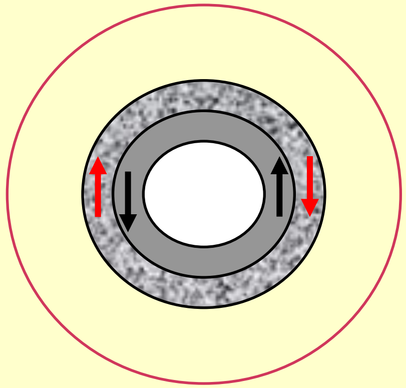
Large scale structures: Definitions

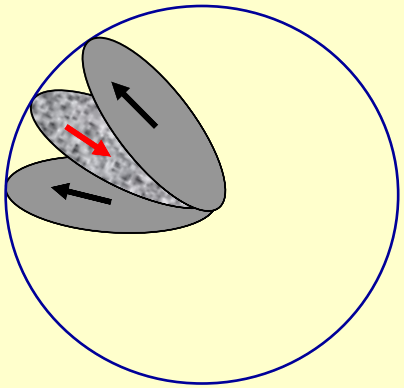
- Zonal flows



- Streamers







Non-linear dynamics

$$\mathbf{v} = -\left(\nabla\phi\times\mathbf{z}\right),\quad \phi = \left\langle\phi\right\rangle + \tilde{\phi} \quad\Rightarrow\quad \mathbf{v} = \left\langle\mathbf{v}\right\rangle + \tilde{\mathbf{v}},\quad \left\langle\tilde{\mathbf{v}}\right\rangle = 0$$

$$\begin{Bmatrix} \nabla n_0 \\ \nabla T_0, \nabla B_0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \textit{drift type} \\ \textit{waves} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \left\langle \tilde{\mathrm{v}}_r \tilde{\mathrm{v}}_\vartheta \right\rangle \\ \text{Reynolds } \textit{stress} \end{Bmatrix} \Rightarrow \frac{\partial\left\langle \mathrm{v}_\vartheta \right\rangle}{\partial t}$$

free thermodynamic energy $\xrightarrow{\text{is converted}}$ via fluctuation induced Reynolds stress $\rightarrow$ kinetic energy of large scale flow (zonal flow)

Non-linear dynamics

$$\mathbf{v} = -\left(\nabla\phi\times\mathbf{z}\right),\quad \phi = \left\langle\phi\right\rangle + \tilde{\phi} \quad\Rightarrow\quad \mathbf{v} = \left\langle\mathbf{v}\right\rangle + \tilde{\mathbf{v}},\quad \left\langle\tilde{\mathbf{v}}\right\rangle = 0$$

$$\left.\begin{array}{l} \mathbf{v} = -\left(\nabla\phi\times\mathbf{z}\right) \\ \nabla\times\mathbf{v} = \nabla^2\phi\mathbf{z} \\ \omega_{\mathbf{k}}\left(\mathbf{k}\right)\neq 0 \end{array}\right\} \quad\Rightarrow\quad \frac{\partial}{\partial t}\left(\nabla^2\phi-\phi\right)-\left[\left(\nabla\phi\times\mathbf{z}\right)\cdot\nabla\right]\left(\nabla^2\phi+\ln\frac{\omega_{ci}}{n_0}\right)=0$$

$$\frac{\partial}{\partial t}\left\langle \mathrm{v}_{\vartheta}\right\rangle = -\left\langle\frac{\partial}{\partial r}\left(\tilde{\mathrm{v}}_r\tilde{\mathrm{v}}_{\vartheta}\right)\right\rangle$$

CONCLUSIONS

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