Space Physics Lecture Notes

From the Course "Rymdfysik I"

Anders I. Eriksson
Department of Astronomy and Space Physics
Uppsala University

October 2002

Lightly edited lecture notes on some contents of the course which are insufficently treated in the course book. Small corrections done 2006-01-30.

Contents

1 What is space physics?

- **When?** Mostly during the satellite era, from about 1960

- **Where?** Mainly, the space between and around the solid bodies in space.

- **How?**
    - Plasma theory
    - In situ (on the spot) measurements of natural processes using spacecraft
    - Experiments in space (strong radio waves, plasma releases from spacecraft etc.)
    - Remote sensing measurements (radar, ionosondes, radio telescopes, optical instruments etc.)

There are no well defined borders to other sciences. Space physics mainly relates to plasma physics, meteorology (in the upper atmosphere), astronomy and astrophysics (at the sun and other stars).

2 A very short tour of the major solar system plasmas

2.1 What is a plasma?

A plasma is a gas of charged particles. As they are charged, electromagnetic fields affect their motion, and therefore the dynamics of the plasma. Also, the charged particles can carry currents creating electromagnetic fields.

2.2 Solar system plasmas

2.2.1 The sun

High temperature $\Rightarrow$ matter is ionized $\Rightarrow$ the sun is a plasma

2.2.2 The solar wind

A wind of charged particles flows out from the sun through the solar system $\Rightarrow$ interplanetary space is filled with a plasma

2.2.3 Ionospheres

Ionizing radiation + atmosphere = ionosphere

- UV-, gamma, and X-radiation from the sun can ionize particles in a planetary atmosphere

- Cosmic radiation can also cause ionization

- When ionizing the gas, the radiation is stopped by the atmosphere and to not penetrate further down. Therefore, only the upper layer of the atmosphere is ionized $\Rightarrow$ plasma. This plasma is called the ionosphere.

All planets with atmospheres has an ionosphere: Venus, Earth, Jupiter, Saturn, Uranus, Neptune. In addition, comets evaporates a gas cloud (because of radiation from the sun) which is partly ionized, causing a cometary ionosphere.

2.2.4 Magnetospheres

Solar wind + planetary magnetic field = magnetosphere
Thus, all magnetized planets have magnetospheres: Mercury, Earth, Mars, Jupiter, Saturn, Uranus, Neptune

- As the particles in the solar wind are charged, their motion is affected by magnetic field from the planets.

- Also, the charged particles in the solar wind can carry a current that can change the magnetic fields.

- The net result is that the solar wind is deviated by the magnetic field of a planet, and that this magnetic field is confined to a region called the magnetosphere.

Near a magnetized planet, space is thus divided into two regions: the high-speed solar wind, to which the planetary magnitic field does not reach, and the magnetosphere, where the planetary magnetic field is confined but the solar wind cannot enter. The boundary between the two regions is known as the magnetopause. This is a first example of how space plasmas become structured into different regions.

Magnetospheres are not empty: they are filled with a plasma, partly from the planetary ionosphere, partly from a fraction of the solar wind that manages to cross the magnetopause and enter the magnetosphere.

3 Plasmas

3.1 Existence of plasma

In a gas in thermal equlibrium at temperature T, the number of neutral molecules n_n and free electrons n_e are related by the Saha equation

$$\frac{n_e}{n_n} = \left(\frac{2\pi KT}{h^2}\right)^{3/4} \frac{1}{\sqrt{n_n}} \exp\left(-\frac{V_i}{KT}\right) \tag{1}$$

where K and h are Boltzmann's and Planck's constants, and V_i is the ionization energy for the neutral particles. For ordinary air at room temperature, one gets a ridiculously small number, $n_e/n_n \sim 10^{-120}$. For the gas to become a plasma, n_e/n_n must obviously reach much higher values. Looking at the Saha equation, we can identify three possibilities:

- High temperature ($KT \sim V_i$). This gives high kinetic energy to the particles, so that molecules may be ionized in collissions.

- Low density. This makes the probability of recombination low: once an atom is ionized, it is hard to find an electron to recombine with in such a way that both energy and momentum are conserved in the recombination.

- Non-equilibrium. In this case, the Saha equation is no longer valid. For space plasmas, collision mean free paths are usually long and collision frequencies low. This means that it takes a very long time for the plasma to come into equilibrium, and many interesting things may happen before the plasma comes to equilibrium.

3.2 Interactions in plasmas

In a gas of neutral particles (henceforth called a "neutral gas"), the particles interact with each other only through collisions.

In a plasma, the particles interact with each other at all times through the electromagnetic forces. Thus, the dynamics of a plasma is inherently more complicated than that of a neutral gas.

3.3 Particle description of plasmas

The equation of motion for a particle k in a classical non-relativistic plasma (Newton's second law) is

$$\frac{d\mathbf{v}_k}{dt} = q_k(\mathbf{E} + \mathbf{v}_k \times \mathbf{B}) + \text{other forces}. \tag{2}$$

The "other forces" may include for example the gravitational force. Thus, if the electromagnetic (EM) fields $\mathbf{E}$ and $\mathbf{B}$ are given, we can, at least in principle, get the particle motion $\mathbf{v}_k(t)$ and position $\mathbf{r}_k(t)$ by integration. To calculate the EM fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, we have Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{6}$$

To solve these, we must know the charge density $\rho(\mathbf{r}, t)$ and the current density $\mathbf{j}(\mathbf{r}, t)$. However, these are given by the particle positions and motions by

$$\rho(\mathbf{r}, t) = \sum_k q_k \, \delta(\mathbf{r} - \mathbf{r}_k(t)) \tag{7}$$

$$\mathbf{j}(\mathbf{r}, t) = \sum_k q_k \, \mathbf{v}_k \, \delta(\mathbf{r} - \mathbf{r}_k(t)) \tag{8}$$

Hence, we have an infinite chain

$$\mathbf{r}, \mathbf{v} \Leftarrow \mathbf{E}, \mathbf{B} \Leftarrow \rho, \mathbf{j} \Leftarrow \mathbf{r}, \mathbf{v} \Leftarrow \ldots \tag{9}$$

The equations above must therefore be solved simultaneously. Note that the number of equations is very large: there is one of equation (2) for each particle, so in any interesting portion of space, the number of equations will be enormous. However, this description of a plasma is used for doing computer simulations of plasma dynamics, where indeed the above equations or some simplifications of them are solved to find the behaviour of the plasma. Such simulations usually includes a few thousand particles, and are normally not fully three dimensional. However, as the computer capacity increases, the simulations will be more and more realistic, and the importance of numerical simulations of plasmas is likely to increase.

3.4   Statistical description of plasmas

Even if we could solve all the equations above, we wouldn't learn much, since we would get far too much information – we are not interested in the positions and motion of every single particle. Rather, we are interested in statistical averages, for example the density of the plasma. Instead of calculating the motion and position of every particle and then averaging the results, we may try to find equations for the statistical quantities themselves. This is the essence of a statistical description of a plasma.

The fundamental statistical quantity is the distribution function $f(\mathbf{r}, \mathbf{v}, t)$. This tells us how many particles with velocity near $\mathbf{v}$ that are present near the location $\mathbf{r}$ at time t. More specifically, the number of particles of species α in the volume $\mathrm{d}^3 r = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$ that have velocities in the intervals $[v_x, v_x + \mathrm{d}v_x]$, $[v_y, v_y + \mathrm{d}v_y]$, $[v_z, v_z + \mathrm{d}v_z]$ is

$$f_\alpha(\mathbf{r}, \mathbf{v}, t) \, \mathrm{d}^3 r \, \mathrm{d}^3 v = f(x, y, z, v_x, v_y, v_z, t) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}v_x \, \mathrm{d}v_y \, \mathrm{d}v_z. \tag{10}$$

Hence, the SI unit of f is s^3/m^6. There is one distribution function for each particle species. In a plasma, there will thus be one distribution function for the electrons, and one for each ion species.

It is possible to construct equations for how the distribution function evolves in time and space due to the influence of electromagnetic fields and other forces. This is the basis for the most advanced plasma theory, called kinetic theory. We will not discuss kinetic theory in this course.

For a gas or plasma in thermodynamic equilibrium, the distribution function is the Maxwell-Boltzmann distribution

$$f(\mathbf{r}, \mathbf{v}) = \left(\frac{m}{2\pi K T} \right)^{3/2} \exp\left(-\frac{\frac{1}{2} m v^2 + U(\mathbf{r})}{KT} \right) \tag{11}$$

where $U(\mathbf{r})$ is the potential energy. The distribution is shown in Figure 1. Plasmas in space are often far from equilibrium, and non-Maxwellian distributions are frequently encountered. Nevertheless, the Maxwell-Boltzmann distribution is often a good approximation.
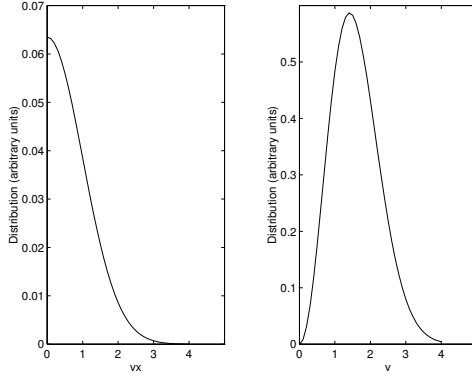
Figure 1: The Maxwell-Boltzmann distribution for a velocity component (left) and for the speed (right). The most probable velocity along any given axis is zero, as seen in the left plot, while the most probable speed is non-zero (right plot).

3.5 Fluid description of plasmas

3.5.1 Fluid parameters

By summing over all velocities, we get a fluid description of a plasma. Here, each particle species is described by what is known as fluid parameters: density, flow speed, temperature and so on. We get the number density by simply summing the distribution function for all velocities:

$$n_\alpha(\mathbf{r}, t) = \int f_\alpha(\mathbf{r}, \mathbf{v}, t) \, \mathrm{d}^3 v. \tag{12}$$

This is the number of particles of species α per unit volume (SI unit: m^{-3}). We get the mass density by multiplying the number of particles per unit volume by the mass of each particle, so

$$\rho_\mathrm{m} = \sum_\alpha m_\alpha n_\alpha, \tag{13}$$

and similarly the charge density is given by

$$\rho = \sum_\alpha q_\alpha n_\alpha. \tag{14}$$

By defining the mean velocity of the particles of species α,

$$\mathbf{v}_\alpha(\mathbf{r}, t) = \frac{1}{n_\alpha} \int \mathbf{v} \, f_\alpha(\mathbf{r}, \mathbf{v}, t) \, \mathrm{d}^3 v, \tag{15}$$

we can also calculate the current density in the plasma as

$$\mathbf{j} = \sum_\alpha q_\alpha n_\alpha \mathbf{v}_\alpha. \tag{16}$$

In this course, we will normally assume that the plasmas we study consist of two particle species: electrons (e) and protons (i). Such a plasma is called a two-component plasma. In this case, we get $\rho_\mathrm{m} = m_\mathrm{e} n_\mathrm{e} + m_\mathrm{i} n_\mathrm{i} \approx m_\mathrm{i} n_\mathrm{i}$ (because $m_e \ll m_i$), $\rho = e(n_\mathrm{i} - n_\mathrm{e})$, and $\mathbf{j} = e(n_\mathrm{i} \mathbf{v}_\mathrm{i} - n_\mathrm{e} \mathbf{v}_\mathrm{e})$.

3.5.2 Fluid in equilibrium

In thermodynamic equilibrium, the plasma follows the Maxwell-Boltzmann distribution (11). The potential energy of a charge in an electrostatic field is $U = q\Phi$, where Φ is the electrostatic potential (SI unit: V). Thus, (11) and (12) implies

$$n_i(\mathbf{r}) = \left(\frac{m}{2\pi KT}\right)^{3/2} \int \exp\left(-\frac{\frac{1}{2}mv^2 + e\Phi(\mathbf{r})}{KT}\right) \, \mathrm{d}^3v = n_0 \, \exp\left(-\frac{e\Phi(\mathbf{r})}{KT}\right) \qquad (17)$$

and, in the same way,

$$n_e(\mathbf{r}) = n_0 \, \exp\left(\frac{e\Phi(\mathbf{r})}{KT}\right). \qquad (18)$$

These equations are known as the Boltzmann relations for ions and electrons. Their basic content is very simple: where the potential energy is low, there will be a lot of particles; where it is high, the particles are scarce. A familiar example where the potential energy is due to gravitation rather than electric effects is the ordinary air around us. The higher the potential energy, i.e. the height ($U = m\,g\,h$ in this case), the lower the density.

3.5.3 Electrostatic (Debye) shielding

A charge $+Q$ will affect the motion of other charges in the plasma. If a particle is negative, its orbit will bend towards Q, if it is positive, the orbit will be bent away from Q. The net result is that negative particles will spend more time near Q and positive particles will spend more time away from Q, thus creating a net negative charge density in space near Q. In this way, the positive charge Q is shielded by a cloud of negative particles. In the same way, a negative particle is shielded by positive particles. This is known as Debye shielding, and is mathematically treated as below.

For an equilibrium plasma, the ions and electrons follows the Boltzmann relations (17) and (18). The charge density (14) in the plasma is then

$$\rho(\mathbf{r}) = e(n_i - n_e) = n_0 \left[\exp\left(-\frac{e\Phi(\mathbf{r})}{KT}\right) - \exp\left(\frac{e\Phi(\mathbf{r})}{KT}\right)\right]. \qquad (19)$$

The electrostatic potential is determined by the charge density through $\mathbf{E} = -\nabla\Phi$ and Gauss' law for the electric field (3). Hence, we have

$$\nabla^2\Phi = -\nabla \cdot \mathbf{E} = -\rho/\epsilon_0 = \frac{n_0 e}{\epsilon_0}\left[\exp\left(\frac{e\Phi}{KT}\right) - \exp\left(-\frac{e\Phi}{KT}\right)\right]. \qquad (20)$$

This is a nonlinear ordinary differential equation, solvable only by numerical methods. However, for the case $e\Phi/KT \ll 1$, we can expand in powers of $e\Phi/KT$ and neglect higher terms in this quantity to get

$$\nabla^2\Phi = \frac{n_0 e}{\epsilon_0}\left[1 + \frac{e\Phi}{KT} + ... - \left(1 - \frac{e\Phi}{KT} + ...\right)\right] \approx \frac{n_0 e^2}{2\epsilon_0 KT}\Phi. \qquad (21)$$

To solve this equation around a charge Q in the plasma, we may assume spherical symmetry and introduce spherical coordinates:

$$\nabla^2\Phi = \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\Phi) \approx \frac{n_0 e^2}{2\epsilon_0 KT}\Phi = \frac{1}{\lambda_D^2}\Phi. \qquad (22)$$
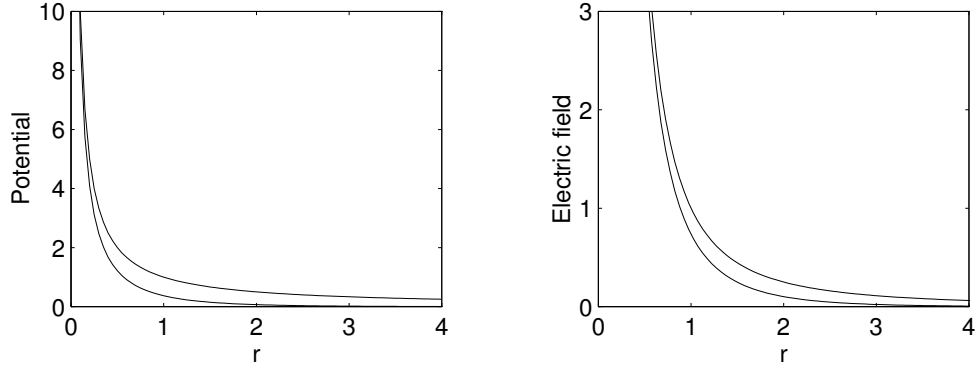
9

Figure 2: Comparison of unscreened Coulomb (upper curves) and screened Debye (lower curves) potential (left plot) and field (right plot). The r axis is in units of the Debye length λ_D; the units on the other axis are arbitrary.

We have here introduced a quantity called the Debye length

$$\lambda_D = \sqrt{\frac{\epsilon_0 KT}{2n_0 e^2}}. \tag{23}$$

The solution of (22) is straightforward, and yields exponential solutions for $r\Phi$. Rejecting the exponentially growing solutions on physical grounds, we are left with

$$\Phi(r) = C \frac{\exp(-r/\lambda_D)}{r}. \tag{24}$$

To determine the constant C, we note that at small values of r, the exponential goes to 1. In this region, the potential should equal the normal Coulomb potential, since for small r, there is very little charge inside r that can do the screening. Hence, we must have $C = 1/(4\pi\epsilon_0)$, and so

$$\Phi(r) = \frac{1}{4\pi\epsilon_0 r} \exp(-r/\lambda_D) \tag{25}$$

A comparison of the unscreened Coulomb potential in vacuum and the screened Debye potential in a plasma is seen in Figure 2. The very rapid decay of the Debye potential above $r = \lambda_D$ ($r = 1$ in the figure) is clearly seen. This has important physical implications: on distances longer than λ_D, the influence of individual particles are unimportant. Only the collective fields created by the "cooperation" of many charges are important. This is very fortunate, since it means that we do not have to keep track of what every particle is doing, but can concentrate on statistical properties like plasma density, velocity etc.

3.5.4 Equation of continuity

All phenomena of interest to us are deviations from perfect thermodynamic equilibrium, small or large. To study such phenomena in a plasma, one needs equations for the ion gas and the electron gas. The fundamental equation describing the behaviour of any gas is the continuity equation,

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = Q - L. \tag{26}$$

10

Here, Q and L are the source and loss densities, respectively (number of particles created or lost in a unit volume during unit time). The physical content of this equation is very simple: a change of the number of particles in a unit volume ($\partial n/\partial t$) is due to flow of particles ($n\mathbf{v}$), creation of particles (Q), and destruction of particles (L). Considering some volume V bounded by a surface S and containing a number of particles N, the net number of particles flowing into V per unit time is

$$\left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{flow}} = - \oint_S n\,\mathbf{v}\,\cdot\,\mathrm{d}\mathbf{S}, \tag{27}$$

using the usual convention that the normal to the surface S is directed outward from V. Using Gauss' theorem, this may be written as

$$\left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{flow}} = - \int_V \nabla \cdot (n\,\mathbf{v})\,\mathrm{d}V. \tag{28}$$

The change in N due to creation and annihilation of particles must be

$$\left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{source}} = \int_V Q\,\mathrm{d}V \tag{29}$$

and

$$\left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{loss}} = - \int_V L\,\mathrm{d}V. \tag{30}$$

The total change in N may clearly be written in two ways:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{flow}} + \left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{source}} + \left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{loss}} \tag{31}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V n\,\mathrm{d}V = \int_V \frac{\partial n}{\partial t}\,\mathrm{d}V. \tag{32}$$

From a comparison of these two expressions, equation (26) follows directly.

3.5.5 Equation of motion

In the fluid picture, the equations of motion for the ion gas and the electron gas are "Newton's second law per unit volume". The forces acting on a volume of gas of charged particles are the electromagnetic forces, the pressure, and possible other forces like gravitation. Thus,

$$n_\alpha m_\alpha \frac{\mathrm{d}\mathbf{v}_\alpha}{\mathrm{d}t} = n_\alpha q_\alpha (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) - \nabla p_\alpha + \text{other force densities}. \tag{33}$$

There is one such equation of motion for each particle species, e.g. for protons and electrons in the two-component plasma. These two equations are known as the two-fluid equations of motion.

The fluid equations of motion (33) looks very similar to the equation of motion (2) for a single particle. Basically, the differences are:

- Equation (2) considers forces [SI unit: N], equation (33) considers forces per unit volume [SI unit: N/m^3].

- Equation (2) considers the exact velocity $\mathbf{v}_{\text{k}}$ of some particular particle k, while equation (33) considers the average velocity $\mathbf{v}_\alpha$ of many particles of the species α.

- The first-order statistical effect of the deviation of the individual particle velocities from the average velocity is included in (33) as the pressure term.

3.5.6 Convective derivative

In equation (33), the derivative $\mathrm{d}/\mathrm{d}t$ is the total time derivative seen by an observer travelling with the fluid velocity $\mathbf{v}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \tag{34}$$

For instance, when entering a warm house on a cold winter day, one feels the temperature changing with time: at the time when you are outside, the temperature you feel is low, an instant later, when you are inside, the temperature is high. By dividing by the time it took you to go inside, you get a derivative $\mathrm{d}T/\mathrm{d}t$. This is completely due to your motion, $\mathbf{v} \cdot \nabla T$, the change of temperature at any fixed point, $\partial T/\partial t$, being essentially zero.

4 Magnetic fields

4.1 The dipole field

In the static case ($\partial/\partial t = 0$), it follows from Maxwell's equations (3) – (6) that the magnetic field $\mathbf{B}$ is governed by the two equations

$$\nabla \cdot \mathbf{B} = 0, \tag{35}$$

sometimes known as Gauss' law for the magnetic field or the condition of no magnetic monopoles, and

$$\nabla \times \mathbf{B} = \mu_0 \, \mathbf{j}, \tag{36}$$

called Ampère's law.

In perfect vacuum, there are no free charges that can carry a current, so here (36) reduces to $\nabla \times \mathbf{B} = 0$. From vector analysis, we know that this is a sufficient condition for the existence of a scalar magnetic potential Ψ such that

$$\mathbf{B} = -\nabla\Psi. \tag{37}$$

Combining with (35), we get a Laplace equation for the magnetic potential:

$$\nabla^2\Psi = -\nabla \cdot (\nabla\Psi) = -\nabla \cdot \mathbf{B} = 0. \tag{38}$$

If all sources of the magnetic field (all currents) are contained inside a sphere of radius $r = a$, then $\mathbf{B}$ can be described by Ψ outside this sphere. From the course in "Mathematical methods of physics", we recall that in spherical coordinates, the general solution of the Laplace equation (38) is the multipole expansion

$$\Psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{A_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi) \tag{39}$$

where we have neglected terms that grow with distance r, as all fields should decrease with distance outside the source sphere $r = a$. The functions Y_{lm} are known as spherical harmonics. This expansion is very nice as it splits the potential into terms with different r-dependence: the higher the l-value, the faster the potential, and thus the field, decays with increasing r. For the case of the magnetic potential the term $l = 0$ will be missing, since this term does not fulfill the requirement that $\nabla \cdot \mathbf{B} = 0$ everywhere, as required by (35). Therefore, the first non-vanishing terms in the expansion (39) are the $l = 1$ terms, which are known as the dipole terms. With a suitable choice of coordinate axis (putting the symmetry axis along the dipole moment $\mathbf{M}$ of the sources $r < a$, i.e. $\mathbf{M} = M\hat{\mathbf{z}}$, only the $m = 0$ term need to be used, so that the first term in the expansion (39) is

$$\Psi = A_{10} \frac{\cos\theta}{r^2}. \tag{40}$$

The coefficient is related to the dipole moment M by $A_{10} = -\mu_0 M/4\pi$, so taking the gradient, we find that the dipole magnetic field is

$$\mathbf{B} = -\nabla\Psi = -\frac{\mu_0 M}{4\pi r^3} \left(2\cos\theta \, \hat{\mathbf{r}} \; + \; \sin\theta \, \hat{\theta} \right) \tag{41}$$

or

$$B_r = -\frac{\mu_0 M}{2\pi r^3} \cos\theta \tag{42}$$

$$B_\theta = -\frac{\mu_0 M}{4\pi r^3} \sin\theta. \tag{43}$$

The field strength $B = |\mathbf{B}|$ thus is

$$B = \sqrt{B_r^2 + B_\theta^2} = \frac{\mu_0 M}{4\pi r^3} \sqrt{4 - 3\sin^2\theta}, \tag{44}$$

going to zero as $1/r^3$ when r increases toward infinity. The fundamental reason why this dipole field is important is simply that it is the term in complete multipole expansion (39) that decay slowest with increasing distance. Sufficiently far away from any source, the dipole field will dominate over the other terms in the multipole expansion (39).

4.2 Field lines

A field line is a curve which everywhere is parallel to some given field. Denoting the field by $\mathbf{B}$, we thus have

$$\mathrm{d}\mathbf{r} = \text{constant} \cdot \mathbf{B} \tag{45}$$

where $\mathrm{d}\mathbf{r}$ is a line element along the field line. In Cartesian coordinates we have $\mathrm{d}\mathbf{r} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}} + \mathrm{d}z\,\hat{\mathbf{z}}$, and hence the field line equation is

$$\frac{\mathrm{d}x}{B_x} = \frac{\mathrm{d}y}{B_y} = \frac{\mathrm{d}z}{B_z}. \tag{46}$$

In spherical coordinates, $\mathrm{d}\mathbf{r} = \mathrm{d}r\,\hat{\mathbf{r}} + r\,\mathrm{d}\theta\,\hat{\theta} + r\,\sin\theta\,\mathrm{d}\phi\,\hat{\phi}$, and thus

$$\frac{\mathrm{d}r}{B_r} = \frac{r\,\mathrm{d}\theta}{B_\theta} = \frac{r\,\sin\theta\,\mathrm{d}\phi}{B_\phi} \tag{47}$$

is the equation of the field lines.

For the dipole field (41), there is no B_ϕ, so the last term in (47) disappears. The remaining equation for the field lines of the dipole field is

$$\frac{\mathrm{d}r}{2\cos\theta} = \frac{r\,\mathrm{d}\theta}{\sin\theta}. \tag{48}$$

This is a spearable ordinary differential equation, which may be written as

$$\frac{\mathrm{d}r}{r} = 2\frac{\cos\theta}{\sin\theta}\mathrm{d}\theta \tag{49}$$

$$\implies \ln r = \ln\sin^2\theta + C \tag{50}$$

$$\implies r = r_0\sin^2\theta. \tag{51}$$

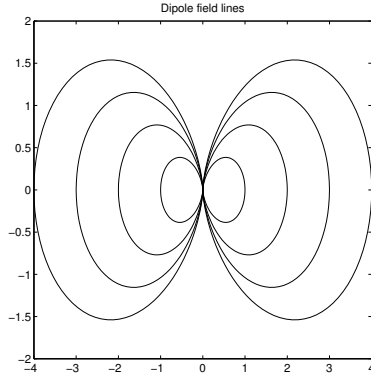This is then the equations of the dipole field lines, some of which are plotted in Figure 3.

14

Figure 3: The field lines from a magnetic dipole at the origin.

4.3 Planetary magnetic fields

Planets with fluid interiors (Mercury, Earth, Jupiter, Saturn, Uranus, Neptune) generate magnetic fields deep in their cores (why and how these fields are generated is briefly discussed in section 4.9 below). On the planetary surfaces, the fields are fairly dipol-like, since other terms in the multipole expansion decays faster with distance from the core. If space around the planets where a vacuum, we would expect that the fields would become more and more dipole-like with increasing distance from the planet, but this is in general not the case. Due to currents generated in the plasma surrounding the planets when it interacts with the solar wind, the magnetic fields are distorted to form magnetospheres.

4.4 Field transformations

The relativistic (Lorentz) transformations of the electromagnetic fields between a system S and another system S' moving with velocity $\mathbf{v}$ as seen by an observer in S are

$$\mathbf{E}'_{\mathrm{T}} = \frac{1}{\sqrt{1 - v^2/c^2}} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \tag{52}$$

$$\mathbf{B}'_{\mathrm{T}} = \frac{1}{\sqrt{1 - v^2/c^2}} \left(\mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \right) \tag{53}$$

for the transverse part of the fields, i.e. the components perpendicular to $\mathbf{v}$, and

$$\mathbf{E}'_{\mathrm{L}} = \mathbf{E}_{\mathrm{L}} \tag{54}$$

$$\mathbf{B}'_{\mathrm{L}} = \mathbf{B}_{\mathrm{L}} \tag{55}$$

for the longitudinal components (along $\mathbf{v}$). In the non-relativistic limit $v/c \longrightarrow 0$ this reduces to the Galilean transformation equations

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{56}$$

$$\mathbf{B}' = \mathbf{B}. \tag{57}$$

The Lorentz force $q\,\mathbf{v} \times \mathbf{B}$ one a charged particle in a magnetic field thus is the force due to the electric field in the reference frame of the particle.

4.5 Frozen-in field lines

The equation of motion (33) for a species alpha of charged particles is

$$n_\alpha m_\alpha \frac{\mathrm{d}\mathbf{v}_\alpha}{\mathrm{d}t} = n_\alpha q_\alpha (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) - \nabla p_\alpha + \text{other force densities}. \tag{58}$$

All terms in this equation except the first on the right hand side contains spatial or temporal derivatives. If these derivatives are small, so that we look at almost stationary (slow) and almost spatially constant (large scale) processes, then the only remaining term is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} \approx 0. \tag{59}$$

Comparing to the Galilean transformation equation (56), we see that this means that the electric field in the plasma frame of reference will be zero. We derived (59) from the equations of motion of the plasma with the assumption of slow and large-scale variations, so the basic physical content of (59) is that given sufficiently long time, the plasma will adjust its flow so as to short-circuit all electric fields in its rest frame.

We will no look at an interesting consequence of equation (59), known as the "frozen-in magnetic field" phenomenon. Consider a closed curve C flowing with the plasma. All parts of the curve are supposed to follow the plasma motion at their particular point, so in general, the shape of the curve as well as its circumference will change with time. The magnetic flux through C is

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{60}$$

where S is a surface whoose boundary is C. The flux Φ will change due to (I) that C will enter regions with different $\mathbf{B}$, and due to (II) that $\mathbf{B}$ changes in time.

The change in Φ due to (I) is calculated as follows. The area covered by a line element $\mathrm{d}\mathbf{l}$ of C due to its motion during a time interval $\mathrm{d}t$ is (Figure 4)

$$\mathrm{d}\mathbf{A} = \mathbf{v} \times \mathrm{d}\mathbf{l}\,\mathrm{d}t. \tag{61}$$

The change in flux through C during dt due to the motion of C is

$$\mathrm{d}\Phi = \mathbf{B} \cdot \mathrm{d}\mathbf{A} = \mathbf{B} \cdot (\mathbf{v} \times \mathrm{d}\mathbf{l}\,\mathrm{d}t) = (\mathbf{B} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{l}\,\mathrm{d}t \tag{62}$$

where we have used a well known vector relation. Summing up all line elements $\mathrm{d}\mathbf{l}$ of C and dividing by $\mathrm{d}t$, we get the total flux change due to the motion of C as

$$\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)_\mathrm{I} = -\oint_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l}. \tag{63}$$

The change in Φ due to (II) is

$$\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)_\mathrm{II} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}. \tag{64}$$

Using the Faraday-Henry law (5) and Stokes' theorem, we get

$$\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)_\mathrm{II} = -\int_S (\nabla \times \mathbf{E}) \cdot \mathrm{d}\mathbf{S} = -\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l}. \tag{65}$$
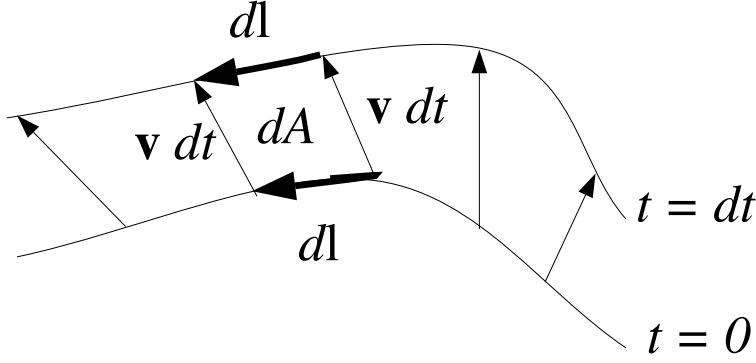
16

Figure 4: The area dA covered in time dt by a line element $d\mathbf{l}$ of a curve which is moving with the flow speed $\mathbf{v}$. The thin arrows mark the displacement $\mathbf{v}\,dt$.

Adding the contributions from (I) and (II) (equations (63) and (65)), we get

$$\frac{d\Phi}{dt} = \left(\frac{d\Phi}{dt}\right)_{\mathrm{I}} + \left(\frac{d\Phi}{dt}\right)_{\mathrm{II}} = -\oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \tag{66}$$

But according to (59), the integrand will be zero for slow and large-scale phenomena. Therefore, for such phenomena, the flux through any closed curve following the plasma flow will be constant. This is known as the "freezing-in" of the magnetic field into the plasma.

It is possible to show that the flux conservation implies that two elements of plasma which at one time are on the same magnetic field line always will be so. It is therefore possible to picture the magnetic field lines as ropes frozen into the plasma and following its motion.

4.6   Energy densities in a plasma

We can write the total energy W in a volume V of plasma as the integral of an energy density w, $W = \int_V w\,dV$. The energy has contributions from four sources. Most obvious is perhaps the thermal energy density due to the random motion of ions and electrons,

$$w_T = \frac{3}{2} n_{\mathrm{i}} K T_{\mathrm{i}} + \frac{3}{2} n_{\mathrm{e}} K T_{\mathrm{e}}. \tag{67}$$

We also have a kinetic energy density due to the ordered motion (flow) of the plasma:

$$w_K = \frac{1}{2} n_{\mathrm{i}} m_{\mathrm{i}} v_{\mathrm{i}}^2 + \frac{1}{2} n_{\mathrm{i}} m_{\mathrm{i}} v_{\mathrm{i}}^2. \tag{68}$$

These are the terms that would appear in an ordinary neutral gas. However, in a plasma one also needs to consider the energy contained in the electromagnetic fields, since this energy can be converted to particle energy and vice versa. Thus, we have to include the energy density due to the existence of a magnetic field:

$$w_B = \frac{1}{2\mu_0} B^2. \tag{69}$$

17

For non-relativistic plasma flows, the electric energy density

$$w_E = \frac{1}{2}\epsilon_0 E^2 \qquad (70)$$

can be neglected. To see this, we consider the ratio $w_E/w_B = \mu_0\epsilon_0 E^2/B^2 = (E/B)^2/c^2$. However, from $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$, we get $E/B \approx v$, where v is the plasma flow speed. Hence we have $w_E/w_B \approx (v/c)^2$, so that the energy in the electric field usually is negligible in comparison to the magnetic energy.

4.7   The interplanetary magnetic field

The relation between the energy densities in a plasma can give us clues to the behaviour of the plasma. For instance, in the solar wind, the kinetic energy density w_K is much higher than w_T and w_B (compare Table 1 on p 94 in Kivelson-Russell). It is therefore reasonable to assume that the solar wind can be described in terms of normal gas flow, without consideration of magnetic effect. Since the magnetic field contains much less energy than the plasma flow, the field can be expected to follow the flow without changing it very much. Why so? Well, think about it this way. Assume $w_K = 100w_B$ That the flow and the B-field interact mean that they exchange energy (and momentum). If the energy density is higher in the flow than in the B-field, a 1 % loss of energy from the flow is a 100 % gain of energy for the field. This means that just a little change of the flow can completely change the field. On the other hand, even a complete annihilation of the B-field will only change the energy in the flow by 1 %. This means that it is the flow that dominates the dynamics, in this case. In some other situation, where $w_B >> w_K$, the opposite would be true.

Thus the solar wind flow determines what the frozen-in magnetic field has to do. As the magnetic field is frozen into the expanding solar wind plasma, magnetic flux is transfered from the sun out into interplanetary space. This is why there is an interplanetary magnetic field with a strength of typically a few nT at Earth orbit. If space was a vacuum, the influence of the solar magnetic should be only through the magnetic dipole field, decaying as $1/r^3$ (equation (41), thereby being completely negligible at Earth orbit.

4.8   Magnetohydrodynamics

In section 3.5, we considered the plasma as consisting of an electron fluid and an ion fluid (or several ion fluids). There is an even simpler description of the plasma, in terms of one conducting fluid. This model is known as magnetohydrodynamics (MHD). Adding the equations of motion for ions and electrons (33), we get the MHD equation of motion

$$\rho_m \frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B} - \nabla p \qquad (71)$$

where ρ_m is the mass density (equation (13)), $p = p_i + p_e$ is the total pressure, and $\mathbf{v}$ is the weighted mean velocity of electrons and ions,

$$(m_i + m_e)\mathbf{v} = m_i\mathbf{v}_i + m_e\mathbf{v}_e. \qquad (72)$$

The magnetohydrodynamical description of a plasma is valid for slow processes on a large scale, so that it is reasonable to assume that the electron and ion fluids have the same number density.

Using Ampère's law (36), the $\mathbf{j} \times \mathbf{B}$ term in the MHD equation of motion (71) may be written

$$\mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \frac{B^2}{2\mu_0} + \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}. \tag{73}$$

The last term on the left hand side is called the magnetic tension, being related to the curvature of magnetic field lines and in some respects similar to the ordinary tension in a string, while the first term is the gradient of a quantity known as the magnetic pressure,

$$p_\mathrm{B} = \frac{B^2}{2\mu_0}. \tag{74}$$

The MHD equation of motion (71) thus can be written as

$$\rho_m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla(p + p_\mathrm{B}) + \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}. \tag{75}$$

This justifies the name magnetic pressure for the term $p_\mathrm{B} = B^2/2\mu_0$, equal to the magnetic energy density.

4.9 Dynamos

Magnetic fields are present almost everywhere in the universe, so there must be some dynamo processes generating them. Such processes are described by the MHD equation of motion (73) together with equation (73), the Faraday-Henry law (5), and the freezing-in condition (59):

$$\rho_m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{76}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{77}$$

Together, these two vector equations form a set of six equations for six unknowns ($\mathbf{v}$ and $\mathbf{B}$). Together with appropriate boundary and initial conditions, this defines the evolution of $\mathbf{v}$ and $\mathbf{B}$ in time and space. In general, other terms would have to be included, like the pressure p neglected in equation (71) and dissipation effects due to viscosity and resistivity, but in principle, these equations holds the key to the dynamo problem. For instance, the plasma flow in the sun must generate the solar magnetic field as described by (77); this magnetic field then affects the flow as described by (76). The MHD equations are applicable not only to a plasma but also to other conducting fluids, like planetary interiors, and therefore equations (76) and (77), or some generalizations thereof, also describes the generation of planetary magnetic fields in terms of the flows inside the planetary cores.

In general, nature gives solutions to (76) and (77) with non-zero magnetic fields: all planets thought to have fluid interiors also have magnetic fields, and so does stars and galaxies. The dynamo equations have been the subject of much study, and some general properties of the system are known. For example, it is known that they have no stationary axisymmetric solution. In this context, it is interesting to note that no planetary magnetic field has its dipole axis perfectly aligned to the rotation axis of the planet. Also, it is known that the Earth's field is non-staionary, having had several pole-reversals with intervals of some tens of thousands of years. For the sun, there is at least one period of much faster change: the sunspot cycle of 22 years.